SCALED ASYMPTOTICS FOR SOME q-SERIES AS q APPROACHING UNIT

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ABSTRACT. In this work we investigate Plancherel-Rotach type asymptotics for some q-series as $q \to 1$. These q-series generalize Ramanujan function $A_q(z)$; Jackson's q-Bessel function $J_\nu^{(2)}(z;q)$, Ismail-Masson orthogonal polynomials(q^{-1}-Hermite polynomials) $h_n(x|q)$, Stieltjes-Wigert orthogonal polynomials $S_n(x;q)$, q-Laguerre orthogonal polynomials $L_n^{(\alpha)}(x;q)$ and confluent basic hypergeometric series.

1. INTRODUCTION

In [7] we derived certain Plancherel-Rotach type asymptotics for some q-series. These q-series generalize Ramanujan's entire function $A_q(z)$, Jackson's q-Bessel function $J_\nu^{(2)}(z;q)$, Ismail-Masson orthogonal polynomials (q^{-1}-Hermite polynomials) $h_n(x|q)$, Stieltjes-Wigert orthogonal polynomials $S_n(x;q)$, q-Laguerre orthogonal polynomials $L_n^{(\alpha)}(x;q)$ and confluent basic hypergeometric series.

In this work we shall employ the method used in [8] to study the scaled asymptotics of these q-series as $q \to 1$. In section §2 we list some common notations from q-series and special functions. We present our results in section §3 and prove them in section §4. Throughout this work we always assume that $0 < q < 1$ unless otherwise stated.

2. PRELIMINARIES

For a complex number z, we define [1, 2, 4, 5]

$$(1) \qquad (z;q)_\infty := \prod_{k=0}^{\infty}(1 - zq^k),$$

and the q-Gamma function is defined as

$$(2) \qquad \Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z} \quad z \in \mathbb{C}.$$

The q-shifted factorials of $a, a_1, \ldots a_m$ are given by

$$(3) \qquad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad (a_1,\ldots,a_m;q)_n := \prod_{k=1}^{m}(a_k;q)_n$$

for all integers $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Given two nonnegative integers s, r and two sets of complex numbers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$, a basic hypergeometric series ${}_s\phi_r$ is formally defined as

$$(4) \qquad {}_s\phi_r \left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q, z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k (-zq^{-\ell})^k q^{\ell k^2}}{(q, b_1, \ldots, b_s; q)_k},$$

where

$$(5) \qquad \ell := \frac{s + 1 - r}{2},$$

and it is a confluent basic hypergeometric series if $\ell > 0$.

Given nonnegative integers r, s, t and a positive number ℓ, we define [7]

$$(6) \qquad g(a_1, \ldots, a_r; b_1, \ldots, b_s; q; \ell; z)$$
$$:= \sum_{k=0}^{\infty} \frac{(q^{k+1}, b_1 q^k, \ldots, b_s q^k; q)_\infty q^{\ell k^2} (-z)^k}{(a_1 q^k, \ldots, a_r q^k; q)_\infty},$$

$$(7) \qquad h(a_1, \ldots, a_r; b_1, \ldots, b_s; c_1, \ldots, c_t; q; \ell; z)$$
$$:= \sum_{k=0}^{n} \frac{(q^{k+1}, b_1 q^k, \ldots, b_s q^k; q)_\infty q^{\ell k^2} (-z)^k}{(a_1 q^k, \ldots, a_r q^k; q)_\infty} \frac{(q, c_1, \ldots, c_t; q)_n}{(q, c_1, \ldots, c_t; q)_{n-k}},$$

where

$$(8) \qquad 0 \le a_1, \ldots, a_r, b_1, \ldots, b_s, c_1, \ldots, c_t < 1.$$

Jackson's q-Bessel function $J_\nu^{(2)}(z; q)$ is defined as [4, 2, 5]

$$(9) \qquad J_\nu^{(2)}(z; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2 + k\nu}(-1)^k}{(q, q^{\nu+1}; q)_k} \left(\frac{z}{2} \right)^{2k+\nu}, \quad \nu > -1.$$

The Ismail-Masson polynomials $\{h_n(x|q)\}_{n=0}^{\infty}$ are defined as [4]

$$(10) \qquad h_n(\sinh \xi | q) := \sum_{k=0}^{n} \frac{(q; q)_n q^{k(k-n)} (-1)^k e^{(n-2k)\xi}}{(q; q)_k (q; q)_{n-k}}.$$

Stieltjes-Wigert orthogonal polynomials $\{S_n(x; q)\}_{n=0}^{\infty}$ are defined as [4]

$$(11) \qquad S_n(x; q) := \sum_{k=0}^{n} \frac{q^{k^2}(-x)^k}{(q; q)_k (q; q)_{n-k}}.$$

The q-Laguerre orthogonal polynomials $\left\{ L_n^{(\alpha)}(x; q) \right\}_{n=0}^{\infty}$ are defined as [4]

$$(12) \qquad L_n^{(\alpha)}(x; q) := \sum_{k=0}^{n} \frac{q^{k^2 + \alpha k}(-x)^k (q^{\alpha+1}; q)_n}{(q; q)_k (q, q^{\alpha+1}; q)_{n-k}}$$

for $\alpha > -1$. Clearly, we have

$$
(13) \qquad A_q(z) = \frac{g(-;-;q;1;z)}{(q;q)_\infty},
$$

$$
(14) \qquad J_\nu^{(2)}(z;q) = \frac{g(-;q^{\nu+1};q;1;z^2 q^\nu/4)}{(q;q)_\infty^2 (2/z)^\nu},
$$

$$
(15) \qquad h_n(\sinh\xi|q) = \frac{h(-;-;-;q;1;e^{-2\xi}q^{-n})}{e^{-n\xi}(q;q)_\infty},
$$

$$
(16) \qquad S_n(x;q) = \frac{h(-;-;-;q;1;x)}{(q;q)_n(q;q)_\infty},
$$

$$
(17) \qquad L_n^{(\alpha)}(x;q) = \frac{h(-;-;q^{\alpha+1};q;1;xq^\alpha)}{(q;q)_n(q;q)_\infty},
$$

$$
(18) \quad {}_s\phi_r\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array} \middle| q,z\right) = \frac{(q,b_1,\ldots,b_s;q)_\infty g(a_1,\ldots,a_r;b_1,\ldots b_s;q;\ell;zq^{-\ell})}{(a_1,\ldots,a_r;q)_\infty}.
$$

The Dedekind $\eta(\tau)$ is defined as [6]

$$
(19) \qquad \eta(\tau) := e^{\pi i\tau/12}\prod_{k=1}^\infty (1 - e^{2\pi ik\tau}),
$$

or

$$
(20) \qquad \eta(\tau) = q^{1/12}(q^2;q^2)_\infty, \quad q = e^{\pi i\tau}, \quad \Im(\tau) > 0.
$$

It has the transformation formula

$$
(21) \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau).
$$

The four Jacobi theta functions are defined as [6]

$$
(22) \qquad \theta_1(v|\tau) := -i\sum_{k=-\infty}^\infty (-1)^k q^{(k+1/2)^2} e^{(2k+1)\pi iv},
$$

$$
(23) \qquad \theta_2(v|\tau) := \sum_{k=-\infty}^\infty q^{(k+1/2)^2} e^{(2k+1)\pi iv},
$$

$$
(24) \qquad \theta_3(v|\tau) := \sum_{k=-\infty}^\infty q^{k^2} e^{2k\pi iv},
$$

$$
(25) \qquad \theta_4(v|\tau) := \sum_{k=-\infty}^\infty (-1)^k q^{k^2} e^{2k\pi iv},
$$

where

$$
(26) \qquad q = e^{\pi i\tau}, \quad \Im(\tau) > 0.
$$

For our convenience, we also use the following notations

$$
(27) \qquad \theta_\lambda(z;q) := \theta_\lambda(v|\tau), \quad z = e^{2\pi iv}, \quad q = e^{\pi i\tau}
$$

with

$$
(28) \qquad \lambda = 1,2,3,4.
$$

The Jacobi's triple product identities are

$$(29) \qquad \theta_1(v|\tau) = 2q^{1/4}\sin\pi v (q^2;q^2)_\infty (q^2 e^{2\pi iv};q^2)_\infty (q^2 e^{-2\pi iv};q^2)_\infty,$$

$$(30) \qquad \theta_2(v|\tau) = 2q^{1/4}\cos\pi v (q^2;q^2)_\infty (-q^2 e^{2\pi iv};q^2)_\infty (-q^2 e^{-2\pi iv};q^2)_\infty,$$

$$(31) \qquad \theta_3(v|\tau) = (q^2;q^2)_\infty (-q e^{2\pi iv};q^2)_\infty (-q e^{-2\pi iv};q^2)_\infty,$$

$$(32) \qquad \theta_4(v|\tau) = (q^2;q^2)_\infty (q e^{2\pi iv};q^2)_\infty (q e^{-2\pi iv};q^2)_\infty.$$

The Jacobi θ functions satisfy transformations

$$(33) \qquad \theta_1\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = -i\sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_1\left(v \mid \tau\right),$$

$$(34) \qquad \theta_2\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_4\left(v \mid \tau\right),$$

$$(35) \qquad \theta_3\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_3\left(v \mid \tau\right),$$

$$(36) \qquad \theta_4\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_2\left(v \mid \tau\right).$$

The Euler Gamma function $\Gamma(z)$ is given by [1, 2, 4, 5]

$$(37) \qquad \frac{1}{\Gamma(z)} := z\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)\left(1+\frac{1}{k}\right)^{-z}, \quad z\in\mathbb{C}.$$

For any real number x, we have

$$(38) \qquad x = \lfloor x\rfloor + \{x\},$$

where the fractional part of x is $\{x\}\in[0,1)$ and $\lfloor x\rfloor\in\mathbb{Z}$ is the greatest integer less or equal x. The arithmetic function

$$(39) \qquad \chi(n) := \begin{cases} 1 & 2\nmid n \\ 0 & 2\mid n \end{cases},$$

which is the principal character modulo 2, satisfies the identities

$$(40) \qquad \chi(n) = 2\left\{\frac{n}{2}\right\} = n - 2\left\lfloor\frac{n}{2}\right\rfloor = \left\lfloor\frac{n+1}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor.$$

Thus,

$$(41) \qquad \left\lfloor\frac{n+1}{2}\right\rfloor = \frac{n+\chi(n)}{2},$$

and

$$(42) \qquad \left\lfloor\frac{n}{2}\right\rfloor = \frac{n-\chi(n)}{2}.$$

3. Main Results

In order to state our results in full generality we also need the following definition:

Definition 3.1. An admissible scale is a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that

$$(43) \qquad \lim_{n\to\infty}\frac{\lambda_n}{\log n} = \infty, \quad \lim_{n\to\infty}\frac{n}{\lambda_n^2} = \infty.$$

Clearly,

$$(44) \qquad \lambda_n = n^\beta \log^\gamma n, \quad \beta + \gamma > 0, \quad 0 < \beta < \frac{1}{2}, \quad \gamma \geq 0,$$

and

$$(45) \qquad \lambda_n = \log^\gamma n, \quad \gamma > 1$$

are admissible scales.

3.1. g-function.

To simplify the type setting in the following theorem, we let

$$(46) \qquad g(z; q) := g(a_1, \ldots, a_r; b_1, \ldots, b_s; q; \ell; z).$$

Theorem 3.2. *Given an admissible scale λ_n, assume that*

$$(47) \qquad z = e^{2\pi v}, \quad q := e^{-\pi \lambda_n^{-1}}, \quad \ell > 0, \quad v \in \mathbb{R},$$

and

$$(48) \qquad a_j := q^{\alpha_j}, \quad b_k := q^{\beta_k}, \quad \alpha_j, \beta_k > 0$$

for

$$(49) \qquad 1 \leq j \leq r, \quad 1 \leq k \leq s.$$

Then,

$$(50) \qquad g(-q^{-4n\ell} z; q) = \exp\left\{ \frac{\pi \lambda_n}{\ell} \left(v + \frac{2n\ell}{\lambda_n} \right)^2 \right\}$$
$$\times \sqrt{\frac{\lambda_n}{\ell}} \left\{ 1 + \mathcal{O}(e^{-\ell^{-1} \pi \lambda_n}) \right\},$$

and

$$(51) \qquad g(q^{-4n\ell} z; q) = \exp\left\{ \frac{\pi \lambda_n}{\ell} \left(v + \frac{2n\ell}{\lambda_n} \right)^2 - \frac{\pi \lambda_n}{4\ell} \right\}$$
$$\times 2\sqrt{\frac{\lambda_n}{\ell}} \left\{ \cos \frac{\pi \lambda_n v}{\ell} + \mathcal{O}\left(e^{-2\ell^{-1} \pi \lambda_n} \right) \right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the Ramanujan's entire function we have:

Corollary 3.3. *Given an admissible scale λ_n, assume that*

$$(52) \qquad z = e^{2\pi v}, \quad q = e^{-\pi \lambda_n^{-1}}, \quad v \in \mathbb{R},$$

we have

$$(53) \qquad A_q(-q^{-4n} z) = \exp\left\{ \pi \lambda_n \left(v + \frac{2n}{\lambda_n} \right)^2 + \frac{\pi \lambda_n}{6} - \frac{\pi}{24\lambda_n} \right\}$$
$$\times \frac{1}{\sqrt{2}} \left\{ 1 + \mathcal{O}\left(e^{-\pi \lambda_n} \right) \right\},$$

and

$$(54) \qquad A_q(q^{-4n}z) = \exp\left\{\pi\lambda_n\left(v + \frac{2n}{\lambda_n}\right)^2 - \frac{\pi\lambda_n}{12} - \frac{\pi}{24\lambda_n}\right\}$$
$$\times \sqrt{2}\left\{\cos\pi\lambda_n v + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the Jackson's q-Bessel function we have:

Corollary 3.4. *For an admissible scale λ_n, assume that*

$$(55) \qquad z = e^{2\pi v}, \quad q = e^{-\pi\lambda_n^{-1}}, \quad v \in \mathbb{R}, \quad \nu > -1,$$

then,

$$(56) \qquad J_\nu^{(2)}(2i\sqrt{zq^{-\nu}}q^{-2n};q) = \frac{\exp\left(\frac{\pi\lambda_n}{3} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n} + \frac{\nu\pi i}{2}\right)}{2\sqrt{\lambda_n}}$$
$$\times \exp\left\{\pi\lambda_n\left(v + \frac{4n+\nu}{2\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}(e^{-\pi\lambda_n})\right\},$$

and

$$(57)$$
$$J_\nu^{(2)}(2\sqrt{zq^{-\nu}}q^{-2n};q) = \frac{\exp\left(\frac{\pi\lambda_n}{12} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n}\right)}{\sqrt{\lambda_n}}$$
$$\times \exp\left\{\pi\lambda_n\left(v + \frac{4n+\nu}{2\lambda_n}\right)^2\right\}\left\{\cos\pi\lambda_n v + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the confluent basic hypergeometric series we have:

Corollary 3.5. *Given an admissible scale λ_n, assume that*

$$(58) \qquad z = e^{2\pi v}, \quad q = e^{-\pi\lambda_n^{-1}}, \quad v \in \mathbb{R},$$

and

$$(59) \qquad \alpha_j, \beta_k > 0, \quad 1 \leq j \leq r, \quad 1 \leq k \leq s.$$

Let

$$(60) \qquad \ell := \frac{s+1-r}{2} > 0, \quad \rho := \sum_{j=1}^{r}\alpha_j - \sum_{j=1}^{s}\beta_j + \ell - 1.$$

Then we have
$$(61)$$
$${}_s\phi_r\left(\begin{array}{c} q^{\alpha_1}, \ldots, q^{\alpha_r} \\ q^{\beta_1}, \ldots, q^{\beta_s} \end{array} \bigg| q, -zq^{-\ell(4n-1)}\right) = \frac{2^\ell \pi^{\rho+\ell} \prod_{j=1}^{r}\Gamma(\alpha_j)}{\sqrt{\ell}\lambda_n^{\rho-1/2}\exp(\ell\pi\lambda_n/3)\prod_{j=1}^{s}\Gamma(\beta_j)}$$
$$\times \left\{\exp\frac{\pi\lambda_n}{\ell}\left(v + \frac{2n\ell}{\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\left(\lambda_n^{-1}\log^2\lambda_n\right)\right\},$$

and

(62)

$$
{}_s\phi_r \left(\begin{array}{c} q^{\alpha_1}, \ldots, q^{\alpha_r} \\ q^{\beta_1}, \ldots, q^{\beta_s} \end{array} \middle| q, zq^{-\ell(4n-1)} \right) = \frac{2^{\ell+1} \pi^{\rho+\ell} \prod_{j=1}^{r} \Gamma(\alpha_j)}{\sqrt{\ell}\,\lambda_n^{\rho-1/2} \exp\left(\frac{\ell\pi\lambda_n}{3} + \frac{\pi\lambda_n}{4\ell} \right) \prod_{j=1}^{s} \beta_j}
$$
$$
\times \left\{ \exp \frac{\pi\lambda_n}{\ell} \left(v + \frac{2n\ell}{\lambda_n} \right)^2 \right\} \left\{ \cos \frac{\pi\lambda_n v}{\ell} + \mathcal{O}\left(\lambda_n^{-1} \log^2 \lambda_n \right) \right\}
$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

3.2. h-**function.** For our convenience we let

(63) $$ h_n(z;q) := h_\ell(a_1, \ldots, a_r; b_1, \ldots, b_s; c_1, \ldots, c_t; q; z). $$

We have similar results for the h function:

Theorem 3.6. *Given an admissible scale λ_n , assume that*

(64) $$ z := e^{2\pi v}, \quad q := e^{-\pi\lambda_n^{-1}}, \quad \ell > 0, \quad v \in \mathbb{R}, $$

and

(65) $$ a_j := q^{\alpha_j}, \quad b_k := q^{\beta_k}, \quad \alpha_j, \beta_k > 0 $$

for

(66) $$ 1 \leq j \leq r, \quad 1 \leq k \leq s. $$

Then,

(67) $$ h(-zq^{-n\ell};q) = \exp\left\{ \frac{\pi\lambda_n}{\ell} \left(v + \frac{\ell(n-\chi(n))}{2\lambda_n} \right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n} \right\} $$
$$
\times \sqrt{\frac{\lambda_n}{\ell}} \left\{ 1 + \mathcal{O}(e^{-\ell^{-1}\pi\lambda_n}) \right\},
$$

and

(68) $$ h(zq^{-n\ell};q) = \exp\left\{ \frac{\pi\lambda_n}{\ell} \left(v + \frac{\ell(n-\chi(n))}{2\lambda_n} \right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi\lambda_n}{4\ell} \right\} $$
$$
\times 2\sqrt{\frac{\lambda_n}{\ell}} \left\{ \cos \frac{\pi\lambda_n}{\ell} \left(v + \frac{\ell(n-\chi(n))}{2\lambda_n} \right) + \mathcal{O}(e^{-2\ell^{-1}\pi\lambda_n}) \right\}
$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For Ismail-Masson orthogonal polynomials we have:

Corollary 3.7. *Given an admissible scale λ_n. For any $v \in \mathbb{R}$ we have*

(69) $$ h_n\left(\sinh \pi \left(v + \frac{i}{2} \right) \middle| q \right) = \frac{\exp\left\{ \frac{\pi n^2}{4\lambda_n} + \frac{\pi\lambda_n}{6} - \frac{\pi(1+12\chi(n))}{24\lambda_n} \right\}}{(-i)^n \sqrt{2}} $$
$$
\times \left\{ \exp\left[\pi\lambda_n \left(v - \frac{\chi(n)}{2\lambda_n} \right)^2 \right] \right\} \left\{ 1 + \mathcal{O}(e^{-\pi\lambda_n}) \right\},
$$

and

(70)

$$h_n(\sinh \pi v \mid q) = (-1)^n \sqrt{2} \exp \left\{ \frac{n^2 \pi}{4\lambda_n} - \frac{(1 + 12\chi(n))\pi}{24\lambda_n} - \frac{\pi \lambda_n}{12} \right\}$$

$$\times \left\{ \exp \left[\pi \lambda_n \left(v - \frac{\chi(n)}{2\lambda_n} \right)^2 \right] \right\} \left\{ \cos \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right) + \mathcal{O}(e^{-2\pi \lambda_n}) \right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For Stieltjes-Wigert orthogonal polynomials we have:

Corollary 3.8. *Given an admissible scale λ_n, assume that*

(71)
$$z := e^{2\pi v}, \quad q := e^{-\pi \lambda_n^{-1}}, \quad v \in \mathbb{R}.$$

Then we have

(72)
$$S_n(-zq^{-n}; q) = \frac{\exp \left\{ \frac{\pi \lambda_n}{3} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n} \right\}}{2\sqrt{\lambda_n}}$$

$$\times \left\{ \exp \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right)^2 \right\} \left\{ 1 + \mathcal{O}\left(e^{-\pi \lambda_n} \right) \right\},$$

and

(73)

$$S_n(zq^{-n}; q) = \frac{\exp \left\{ \frac{\pi \lambda_n}{12} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n} \right\}}{\sqrt{\lambda_n}}$$

$$\times \left\{ \exp \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right)^2 \right\} \left\{ \cos \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right) + \mathcal{O}\left(e^{-2\pi \lambda_n} \right) \right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the q-Laguerre orthogonal polynomials we have:

Corollary 3.9. *Given an admissible scale λ_n, assume that*

(74)
$$z := e^{-2\pi v}, \quad q := e^{-\pi \lambda_n^{-1}}, \quad v \in \mathbb{R}, \quad \alpha > -1.$$

Then we have

(75)
$$L_n^{(\alpha)}(-zq^{-\alpha-n}; q) = \frac{\exp \left\{ \frac{\pi \lambda_n}{3} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n} \right\}}{2\sqrt{\lambda_n}}$$

$$\times \left\{ \exp \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right)^2 \right\} \left\{ 1 + \mathcal{O}\left(e^{-\pi \lambda_n} \right) \right\},$$

and

(76)

$$L_n^{(\alpha)}(zq^{-\alpha-n}; q) = \frac{\exp \left\{ \frac{\pi \lambda_n}{12} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n} \right\}}{\sqrt{\lambda_n}}$$

$$\times \left\{ \exp \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right)^2 \right\} \left\{ \cos \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right) + \mathcal{O}\left(e^{-2\pi \lambda_n} \right) \right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

Remark 3.10. Similar results hold for general τ and β defined in [7] and their proofs are also similar to the proofs for the special cases here. The formulas for the general τ and β may be applicable to the studies in phase transitions and critical phenomena in physics. However, we feel that the formulas for the special cases are more appealing and thus skip the general formuas.

4. PROOFS

The following lemma is from [7], we won't reproduce its proof here.

Lemma 4.1. *Given a complex number a, assume that*

$$(77) \qquad 0 < \frac{|a| \, q^n}{1 - q} < \frac{1}{2}$$

for some positive integer n. Then,

$$(78) \qquad \frac{(a;q)_\infty}{(a;q)_n} = (aq^n; q)_\infty := 1 + r_1(a; n)$$

with

$$(79) \qquad |r_1(a; n)| \le \frac{2 \, |a| \, q^n}{1 - q}$$

and

$$(80) \qquad \frac{(a;q)_n}{(a;q)_\infty} = \frac{1}{(aq^n; q)_\infty} := 1 + r_2(a; n)$$

with

$$(81) \qquad |r_2(a; n)| \le \frac{2 \, |a| \, q^n}{1 - q}.$$

We also need the following lemma:

Lemma 4.2. *Given a sequence of positive numbers $\{\lambda_n\}_{n=1}^\infty$, let*

$$(82) \qquad q = e^{-\pi \lambda_n^{-1}}, \quad \lim_{n \to \infty} \lambda_n = \infty,$$

then,

$$(83) \qquad (q;q)_\infty = \sqrt{2\lambda_n} \exp\left(\frac{\pi}{24\lambda_n} - \frac{\pi\lambda_n}{6}\right) \left\{1 + \mathcal{O}(e^{-4\pi\lambda_n})\right\}$$

and

$$(84) \qquad \frac{1}{(q;q)_\infty} = \frac{\exp\left(\frac{\pi\lambda_n}{6} - \frac{\pi}{24\lambda_n}\right)}{\sqrt{2\lambda_n}} \left\{1 + \mathcal{O}(e^{-4\pi\lambda_n})\right\}$$

as $n \to \infty$.

Proof. From formulas (19), (21) and (21) we get

$$(85) \qquad (q;q)_\infty = \exp\left(\frac{\pi}{24\lambda_n}\right)\eta\left(\frac{i}{2\lambda_n}\right)$$

$$= \sqrt{2\lambda_n}\exp\left(\frac{\pi}{24\lambda_n}\right)\eta(2\lambda_n i)$$

$$= \sqrt{2\lambda_n}\exp\left(\frac{\pi}{24\lambda_n} - \frac{\pi\lambda_n}{6}\right)\prod_{k=1}^{\infty}(1 - e^{-4\pi k\lambda_n}).$$

For sufficiently large n satisfying

$$(86) \qquad \exp(-4\pi\lambda_n) < \frac{1}{3},$$

we have

$$(87) \qquad \prod_{k=1}^{\infty}(1 - e^{-4\pi k\lambda_n}) = 1 + \mathcal{O}(e^{-4\pi\lambda_n}),$$

and

$$(88) \qquad \frac{1}{\prod_{k=1}^{\infty}(1 - e^{-4\pi k\lambda_n})} = 1 + \mathcal{O}(e^{-4\pi\lambda_n})$$

by Lemma 4.1. Consequently,

$$(89) \qquad (q;q)_\infty = \sqrt{2\lambda_n}\exp\left(\frac{\pi}{24\lambda_n} - \frac{\pi\lambda_n}{6}\right)\left\{1 + \mathcal{O}(e^{-4\pi\lambda_n})\right\}$$

$$(90) \qquad \frac{1}{(q;q)_\infty} = \frac{\exp\left(\frac{\pi\lambda_n}{6} - \frac{\pi}{24\lambda_n}\right)}{\sqrt{2\lambda_n}}\left\{1 + \mathcal{O}(e^{-4\pi\lambda_n})\right\}$$

as $n \to \infty$. $\qquad\qquad\square$

Lemma 4.3. *Given a sequence of positive numbers $\{\lambda_n\}_{n=1}^{\infty}$, let*

$$(91) \qquad q = e^{-\pi\lambda_n^{-1}}, \qquad \lim_{n\to\infty}\lambda_n = \infty, \qquad x > 0,$$

then

$$(92) \qquad (q^x;q)_\infty = \frac{\sqrt{2}\pi^{1-x}\lambda_n^{x-1/2}}{\Gamma(x)\exp(\pi\lambda_n/6)}\left\{1 + \mathcal{O}\left(\lambda_n^{-1}\log^2\lambda_n\right)\right\}$$

as $n \to \infty$.

Proof. From the definition of $\Gamma_q(x)$ we have

$$(93) \qquad (q^x;q)_\infty = \frac{(q;q)_\infty(1-q)^{1-x}}{\Gamma_q(x)}.$$

Clearly,

$$(94) \qquad (1-q)^{1-x} = \left(\frac{\pi}{\lambda_n}\right)^{1-x}\left\{1 + \mathcal{O}\left(\lambda_n^{-1}\right)\right\}$$

as $n \to \infty$. In [9] we proved that for $\Re(z) > 0$

$$(95) \qquad \Gamma_q(z) = \Gamma(z)\left\{1 + \mathcal{O}\left((1-q)\log^2(1-q)\right)\right\},$$

as $q \to 1$, thus for $x > 0$,

$$(96) \qquad \Gamma_q(x) = \Gamma(x)\left\{1 + \mathcal{O}\left(\lambda_n^{-1}\log^2\lambda_n\right)\right\}$$

as $n \to \infty$. Combine these equations with equation (83) to obtain the equation (92).

Take $\lambda = 0, \quad \tau = 2, \quad m = 2n$ in Theorem 2.2 of [8] we get the following result: $\qquad\square$

Lemma 4.4. *Assume that* $z \in \mathbb{C}\backslash\{0\}$, $\ell > 0$ *and* (30), *we have*

$$(97) \qquad g(q^{-4n\ell}z; q) = z^{2n}q^{-4n^2\ell}\left\{\theta_4\left(z^{-1}; q^\ell\right) + r_g(n|1)\right\},$$

and

$$(98) \qquad |r_g(n|1)| \leq \frac{2^{s+r+3}\theta_3\left(|z|^{-1}; q^\ell\right)}{(a_1, \ldots, a_r; q)_\infty}\left\{\frac{q^{n+1}}{1-q} + \frac{q^{\ell n^2}}{|z|^n}\right\}$$

for n *sufficiently large. In particular we have*

$$(99)$$
$${}_s\phi_r\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q, zq^{-4n\ell}\right) = \frac{(q, b_1, \ldots, b_s; q)_\infty z^{2n}\left\{\theta_4\left(z^{-1}q^\ell; q^\ell\right) + r_\phi(n|1)\right\}}{(a_1, \ldots, a_r; q)_\infty q^{2\ell n(2n+1)}},$$

and

$$(100) \qquad |r_\phi(n|1)| \leq \frac{2^{s+r+3}\theta_3\left(|z|^{-1}q^\ell; q^\ell\right)}{(a_1, \ldots, a_r; q)_\infty}\left\{\frac{q^{n+1}}{1-q} + \frac{q^{\ell n^2 + \ell n}}{|z|^n}\right\}$$

for n *sufficiently large, where*

$$(101) \qquad \ell := = \frac{s+1-r}{2} > 0.$$

Similarly, if we take $\lambda = 0$ and $\tau = \frac{1}{2}$ in Theorem 2.4 of [8] we get

Lemma 4.5. *Assume that* $z \in \mathbb{C}\backslash\{0\}$, $\ell > 0$, *we have*

$$(102) \qquad h_n(zq^{-n\ell}; q) = (-z)^{\lfloor n/2 \rfloor}q^{-\ell[n^2-\chi(n)]/4}\left\{\theta_4\left(z^{-1}; q^\ell\right) + r_h(n|1)\right\},$$

and

$$(103) \quad |r_h(n|1)| \leq \frac{2^{s+r+2t+5}\theta_3(|z|^{-1}; q^\ell)}{(a_1, \ldots, a_r; q)_\infty}\left\{\frac{q^{\lfloor n/4 \rfloor + 1}}{1-q} + |z|^{\lfloor n/4 \rfloor}q^{\ell\lfloor n/4 \rfloor^2} + \frac{q^{\ell\lfloor n/4 \rfloor^2}}{|z|^{\lfloor n/4 \rfloor}}\right\}$$

for n *sufficiently large.*

4.1. **Proof for Theorem 3.2.** From (24) and (35) to obtain

$$(104) \qquad \theta_3(e^{-2\pi v}; e^{-\pi\ell\lambda_n^{-1}}) = \theta_3\left(vi|\ell\lambda_n^{-1}i\right)$$
$$= \sqrt{\frac{\lambda_n}{\ell}}e^{\pi\ell^{-1}\lambda_n v^2}\theta_3\left(\frac{\lambda_n v}{\ell} \middle| \frac{\lambda_n i}{\ell}\right)$$
$$= \sqrt{\frac{\lambda_n}{\ell}}e^{\pi\ell^{-1}\lambda_n v^2}\left\{1 + \mathcal{O}(e^{-\ell^{-1}\pi\lambda_n})\right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for all $v \in \mathbb{R}$. Clearly we have

$$(105) \qquad \frac{q^{n+1}}{1-q} + q^{\ell n^2}e^{-2n\pi v} = \mathcal{O}(\lambda_n e^{-\pi n\lambda_n^{-1}})$$

as $n \to \infty$, and it is uniform for v in any compact subset of $\mathbb{R}$. From Lemma 4.3 we have

$$(106) \qquad (q^{\alpha_1}, \ldots, q^{\alpha_r}; q)_\infty = \frac{2^{r/2} \pi^{r - \sum_{j=1}^r \alpha_j} \left\{ 1 + \mathcal{O}(\lambda_n^{-1} \log^2 \lambda_n) \right\}}{e^{r\pi\lambda_n/6} \lambda_n^{r/2 - \sum_{j=1}^r \alpha_j} \prod_{j=1}^r \Gamma(\alpha_j)}$$

as $n \to \infty$. Condition (43) gives

$$(107) \qquad g(-q^{-4n\ell} z; q) = \sqrt{\frac{\lambda_n}{\ell}} \exp\left\{ \frac{\pi \lambda_n}{\ell} \left(v + \frac{2n\ell}{\lambda_n} \right)^2 \right\} \left\{ 1 + \mathcal{O}(e^{-\ell^{-1} \pi \lambda_n}) \right\}$$

as $n \to \infty$, and it is uniform for v in any compact subset of $\mathbb{R}$.

Since

$$(108) \qquad \theta_4\left(z^{-1}; q^\ell \right) = \theta_4\left(e^{-2\pi v}; e^{-\ell \pi \lambda_n^{-1}} \right)$$

$$= \theta_4\left(vi \mid \frac{\ell i}{\lambda_n} \right)$$

$$= \sqrt{\frac{\lambda_n}{\ell}} e^{\pi \ell^{-1} \lambda_n v^2} \theta_2\left(\frac{\lambda_n v}{\ell} \mid \frac{i \lambda_n}{\ell} \right)$$

$$= 2\sqrt{\frac{\lambda_n}{\ell}} \exp\left(\frac{\pi \lambda_n v^2}{\ell} - \frac{\pi \lambda_n}{4\ell} \right)$$

$$\times \cos \frac{\pi \lambda_n v}{\ell} \left\{ 1 + \mathcal{O}\left(e^{-2\pi \ell^{-1} \lambda_n} \right) \right\},$$

as $n \to \infty$ and its uniformly in $v \in \mathbb{R}$. Thus we have

$$(109) \qquad g(q^{-4n\ell} z; q) = \exp\left\{ \frac{\pi \lambda_n}{\ell} \left(v + \frac{2n\ell}{\lambda_n} \right)^2 - \frac{\pi \lambda_n}{4\ell} \right\}$$

$$\times 2\sqrt{\frac{\lambda_n}{\ell}} \left\{ \cos \frac{\pi \lambda_n v}{\ell} + \mathcal{O}\left(e^{-2\ell^{-1} \pi \lambda_n} \right) \right\}$$

as $n \to \infty$, and it is uniform on any compact subset of $\mathbb{R}$.

4.2. **Proof for Corollary 3.3.** By Lemma 4.2 and Theorem 3.2 we have

$$(110) \qquad A_q(-q^{-4n} z) = \frac{g(-; -; q; 1; -q^{-4n} z)}{(q; q)_\infty}$$

$$= \exp\left\{ \pi \lambda_n \left(v + \frac{2n}{\lambda_n} \right)^2 + \frac{\pi \lambda_n}{6} - \frac{\pi}{24 \lambda_n} \right\}$$

$$\times \frac{1}{\sqrt{2}} \left\{ 1 + \mathcal{O}\left(e^{-\pi \lambda_n} \right) \right\},$$

and

$$(111) \qquad A_q(q^{-4n} z) = \frac{g(-; -; q; 1; q^{-4n} z)}{(q; q)_\infty}$$

$$= \exp\left\{ \pi \lambda_n \left(v + \frac{2n}{\lambda_n} \right)^2 - \frac{\pi \lambda_n}{12} - \frac{\pi}{24 \lambda_n} \right\}$$

$$\times \sqrt{2} \left\{ \cos \pi \lambda_n v + \mathcal{O}\left(e^{-2\pi \lambda_n} \right) \right\}$$

as $n \to \infty$, it is uniform on any compact subset of $\mathbb{R}$.

4.3. **Proof for Corollary 3.4.** Apply Lemma 4.2 and Theorem 3.2 to equation (14) to get

$$(112) \qquad J_\nu^{(2)}(2i\sqrt{zq^{-\nu}}q^{-2n};q) = \frac{g(-;q^{\nu+1};q;1;-zq^{-4n})}{(q;q)_\infty^2 (i\sqrt{zq^{-\nu}}q^{-2n})^{-\nu}}$$

$$= \frac{\exp\left(\frac{\pi\lambda_n}{3} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n} + \frac{\nu\pi i}{2}\right)}{2\sqrt{\lambda_n}}$$

$$\times \exp\left\{\pi\lambda_n\left(v + \frac{4n+\nu}{2\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}(e^{-\pi\lambda_n})\right\},$$

and

(113)

$$J_\nu^{(2)}(2\sqrt{zq^{-\nu}}q^{-2n};q) = \frac{g(-;q^{\nu+1};q;1;zq^{-4n})}{(q;q)_\infty^2 (\sqrt{zq^{-\nu}}q^{-2n})^{-\nu}}$$

$$= \frac{\exp\left(\frac{\pi\lambda_n}{12} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n}\right)}{\sqrt{\lambda_n}}$$

$$\times \exp\left\{\pi\lambda_n\left(v + \frac{4n+\nu}{2\lambda_n}\right)^2\right\}\left\{\cos\pi\lambda_n v + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, it is uniform on any compact subset of $\mathbb{R}$.

4.4. **Proof for Corollary 3.5.** Apply Lemma 4.2 , Lemma 4.3 and Theorem 3.2 to equation (14) to get

(114)

$${}_s\phi_r\left(\begin{array}{c} q^{\alpha_1}, \ldots, q^{\alpha_r} \\ q^{\beta_1}, \ldots, q^{\beta_s} \end{array} \Big| q, -zq^{-\ell(4n-1)}\right) = \frac{(q, q^{\beta_1}, \ldots, q^{\beta_s}; q)_\infty g(-q^{-4n\ell}z; q)}{(q^{\alpha_1}, \ldots, q^{\alpha_r}; q)_\infty}$$

$$= \frac{(2\pi)^\ell \pi^\rho \prod_{j=1}^r \Gamma(\alpha_j)}{\sqrt{\ell}\lambda_n^{\rho-1/2} \exp(\ell\pi\lambda_n/3) \prod_{j=1}^s \Gamma(\beta_j)}$$

$$\times \left\{\exp\frac{\pi\lambda_n}{\ell}\left(v + \frac{2n\ell}{\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\left(\lambda_n^{-1}\log^2\lambda_n\right)\right\},$$

and

(115)

$${}_s\phi_r\left(\begin{array}{c} q^{\alpha_1}, \ldots, q^{\alpha_r} \\ q^{\beta_1}, \ldots, q^{\beta_s} \end{array} \Big| q, zq^{-\ell(4n-1)}\right) = \frac{(q, q^{\beta_1}, \ldots, q^{\beta_s}; q)_\infty g(q^{-4n\ell}z; q)}{(q^{\alpha_1}, \ldots, q^{\alpha_r}; q)_\infty}$$

$$= \frac{2^{\ell+1}\pi^{\rho+\ell} \prod_{j=1}^r \Gamma(\alpha_j)}{\sqrt{\ell}\lambda_n^{\rho-1/2} \exp\left(\frac{\ell\pi\lambda_n}{3} + \frac{\pi\lambda_n}{4\ell}\right) \prod_{j=1}^s \beta_j}$$

$$\times \left\{\exp\frac{\pi\lambda_n}{\ell}\left(v + \frac{2n\ell}{\lambda_n}\right)^2\right\}\left\{\cos\frac{\pi\lambda_n v}{\ell} + \mathcal{O}\left(\lambda_n^{-1}\log^2\lambda_n\right)\right\}$$

4.5. **Proof for Theorem 3.6.** Clearly,

$$(116) \qquad \frac{q^{\lfloor n/4 \rfloor + 1}}{1-q} + |z|^{\lfloor n/4 \rfloor} q^{\ell \lfloor n/4 \rfloor^2} + \frac{q^{\ell \lfloor n/4 \rfloor^2}}{|z|^{\lfloor n/4 \rfloor}} = \mathcal{O}\left(e^{-\pi n/(4\lambda_n)}\right)$$

as $n \to \infty$ and it is uniformly on any compact subset of $\mathbb{R}$. From equations (42), (104), (106) and (116) we get

$$(117) \qquad h(-zq^{-n\ell}; q) = \exp\left\{ \frac{\pi\lambda_n}{\ell}\left(v + \frac{\ell(n - \chi(n))}{2\lambda_n}\right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n}\right\}$$
$$\times \sqrt{\frac{\lambda_n}{\ell}}\left\{1 + \mathcal{O}(e^{-\ell^{-1}\pi\lambda_n})\right\}$$

as $n \to \infty$, it is uniform on any compact subset of $\mathbb{R}$. Using equations (42), (104), (106), (108) and (116) we get

$$(118) \qquad h(zq^{-n\ell}; q) = \exp\left\{ \frac{\pi\lambda_n}{\ell}\left(v + \frac{\ell(n - \chi(n))}{2\lambda_n}\right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi\lambda_n}{4\ell}\right\}$$
$$\times 2\sqrt{\frac{\lambda_n}{\ell}}\left\{\cos\frac{\pi\lambda_n}{\ell}\left(v + \frac{\ell(n - \chi(n))}{2\lambda_n}\right) + \mathcal{O}(e^{-2\ell^{-1}\pi\lambda_n})\right\}$$

as $n \to \infty$, and it is uniform on any compact subset of $\mathbb{R}$.

4.6. **Proof for Corollary 3.7.** For $v \in \mathbb{R}$, formula 15, Lemma 4.2 and Theorem 3.6 implies

$$(119) \quad h_n\left(\sinh\pi\left(v + \frac{i}{2}\right) \mid q\right) = \frac{h(-;-;-;q;1;-e^{2\pi v}q^{-n})}{(-i)^n e^{n\pi v}(q;q)_\infty}$$
$$= \frac{\exp\left\{\frac{\pi n^2}{4\lambda_n} + \frac{\pi\lambda_n}{6} - \frac{\pi(1+12\chi(n))}{24\lambda_n}\right\}}{(-i)^n\sqrt{2}}$$
$$\times \left\{\exp\left[\pi\lambda_n\left(v - \frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\}\left\{1 + \mathcal{O}(e^{-\pi\lambda_n})\right\},$$

and

(120)

$$h_n(\sinh\pi v \mid q) = \frac{h(-;-;-;q;1;e^{2\pi v}q^{-n})}{(-1)^n e^{n\pi v}(q;q)_\infty}$$
$$= (-1)^n\sqrt{2}\exp\left\{\frac{n^2\pi}{4\lambda_n} - \frac{(1 + 12\chi(n))\pi}{24\lambda_n} - \frac{\pi\lambda_n}{12}\right\}$$
$$\times \left\{\exp\left[\pi\lambda_n\left(v - \frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\}\left\{\cos\pi\lambda_n\left(v + \frac{n - \chi(n)}{2\lambda_n}\right) + \mathcal{O}(e^{-2\pi\lambda_n})\right\}$$

as $n \to \infty$, it is uniform on any compact subset of $\mathbb{R}$.

4.7. **Proof for Corollary 3.8.** From Lemma4.1 and Lemma4.2 we have

$$(121) \qquad \frac{1}{(q;q)_n(q;q)_\infty} = \frac{1}{(q;q)_\infty^2}\frac{(q;q)_\infty}{(q;q)_n}$$
$$= \frac{\exp\left\{\frac{\pi\lambda_n}{3} - \frac{\pi}{12\lambda_n}\right\}}{2\lambda_n}\left\{1 + \mathcal{O}\left(e^{-4\pi\lambda_n}\right)\right\}$$

as $n \to \infty$. Hence, equation (16) and Theorem 3.6 implies

$$(122) \qquad S_n(-zq^{-n};q) = \frac{h(-;-;-;q;1;-zq^{-n})}{(q;q)_n(q;q)_\infty}$$
$$= \frac{\exp\left\{\frac{\pi\lambda_n}{3} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\right\}}{2\sqrt{\lambda_n}}$$
$$\times \left\{\exp\pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\},$$

and

$$(123)$$

$$S_n(zq^{-n};q) = \frac{h(-;-;-;q;1;zq^{-n})}{(q;q)_n(q;q)_\infty}$$
$$= \frac{\exp\left\{\frac{\pi\lambda_n}{12} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\right\}}{\sqrt{\lambda_n}}$$
$$\times \left\{\exp\pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{\cos\pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right) + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, it is uniform on any compact subset of $\mathbb{R}$.

4.8. **Proof for Corollary 3.9.** The equations (17) , (121) and Theorem 3.6 gives

$$(124) \qquad L_n^{(\alpha)}(-zq^{-\alpha-n};q) = \frac{h(-;-;q^{\alpha+1};q;1;-zq^{-n})}{(q;q)_n(q;q)_\infty}$$
$$= \frac{\exp\left\{\frac{\pi\lambda_n}{3} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\right\}}{2\sqrt{\lambda_n}}$$
$$\times \left\{\exp\pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\},$$

and
(125)

$$
\begin{aligned}
L_n^{(\alpha)}(zq^{-\alpha-n};q) &= \frac{h(-;-;q^{\alpha+1};q;1;zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\left\{\frac{\pi\lambda_n}{12} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\right\}}{\sqrt{\lambda_n}} \\
&\times \left\{\exp\pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{\cos\pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right) + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}
\end{aligned}
$$

as $n \to \infty$, and it is uniform for any compact subset of $\mathbb{R}$.

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